PHYSICAL REVIEW VOLUME 110, NUMBER 1 APRIL 1, 1958

Scattering of Light by Protons

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(Received December 12, 1957)

Within the framework of the Chew-Low-Wick development an analysis of the scattering of photons from a nucleon is carried out. It is shown that an exact relationship exists between the Compton effect amplitude and the experimental meson-nucleon scattering phase shifts for all multipoles except magnetic dipole and electric quadrupole provided that effects arising from currents inside the nucleon source (i.e., line currents) are slowly varying functions of photon energy. That part of the magnetic dipole scattering which can be described in terms of the isotopic vector part of the anomalous magnetic moments of the nucleon is also treated exactly. The cross section for the Compton process is evaluated on the basis of the electric and magnetic dipole contributions only, since a nonrecoil theory is clearly expected to be poor for photon energies greater than 300 Mev. Fairly good agreement with experiment is achieved.

I. INTRODUCTION

ONE of the most useful applications of the Chew-Low[1] formalism has been in the derivation of relationships between various processes involving mesons, nucleons, and photons. It has proved possible, using the assumption of a static nucleon, to relate certain parts of the matrix element for single meson photoproduction,[2] low-energy double meson photoproduction,[3] anomalous magnetic moments, and other structural properties of the nucleon[4] to the meson nucleon scattering phase shifts. In this paper a similar relationship is derived for photon nucleon scattering for all multipoles except magnetic dipole and electric quadrupole. It should be emphasized that our calculation is not based on a "one-meson" approximation, but that we have considered the effects of all "n mesons" in the intermediate states.

The assumptions involved in deducing this relationship are the usual ones of the static theory. We describe the nucleon by an extended source function $\rho(x)$ with Fourier transform $v(k)$, and assume the interaction between this source and the meson field is linear in the field, thus coupling only P-wave mesons. The result is derived only to lowest order in the electromagnetic field. The principal assumption which is made is the absence of interaction between the nucleon and anything but P-wave mesons. Thus for any multipole other than $M1$ or $E2$, the meson absorbing the incident photon through the meson current, or through the interaction current, must also be the meson which emits the final photon. As a result, one may establish a correspondence between the diagrams of Fig. 1(a) and those of Fig. 1(b). On the other hand, if the meson upon which the incident photon is absorbed is allowed to interact before emitting the final photon, the correspondence breaks down. In order to interact, the meson must be P wave, and the photon must be magnetic dipole or electric quadrupole. Consequently, different techniques are required to handle these multipoles. The greatest part of the magnetic dipole contribution is trivially related to the scattering amplitude by observing that the isotopic vector anomalous moment absorption of a photon of momentum $\mathbf{k}$ and polarization $\mathbf{e}$ is identical with the absorption of a neutral meson of momentum $(\mathbf{k}\times\mathbf{e})$.

The most unpleasant feature of the static theory in its application to the Compton effect is in the nonlocality of the interaction, expressed by the presence of the cutoff $v(k)$. As has been discussed often before[5]

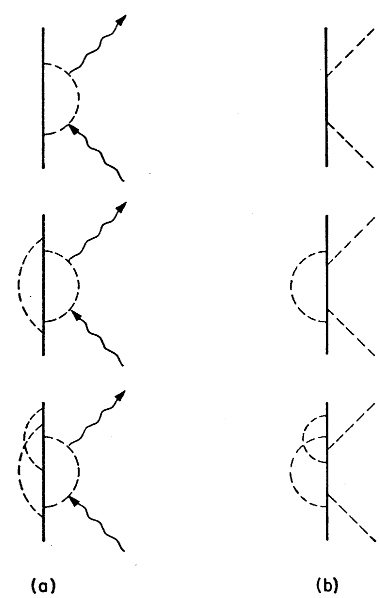

FIG. 1. Diagrams demonstrating the correspondence between (a) Compton scattering and (b) pion-nucleon scattering.

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[1] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956); G. C. Wick, Revs. Modern Phys. **27**, 339 (1955).
[2] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1580 (1956).
[3] R. E. Cutkosky and F. Zachariasen, Phys. Rev. **103**, 1180 (1956).
[4] H. Miyazawa, Phys. Rev. **101**, 1560 (1956).
[5] R. H. Capps and W. G. Holladay, Phys. Rev. **99**, 931 (1955).

the nonlocality produces a violation of gauge invariance unless some mechanism describing the transfer of charge from the nucleon core to the point of emission of the meson is put in. This is usually done by introducing line currents.[5] These serve to restore gauge invariance, but also produce extra electromagnetic interactions which cannot be neglected. It is known[6] that gauge invariance determines the coefficients of the two leading terms in the expansion of the scattering amplitude in powers of the photon energy, the first of these being just the usual Thomson amplitude. These coefficients are not reproduced if the line currents are neglected.

The fact that the line currents are not unique means that it is pointless to attempt to calculate their effects explicitly. Our approach will be to use the fact that we know what the two leading terms will be if gauge invariance is maintained, and assume that the only important effect of the line currents is to produce these terms. Any further contribution of line currents must be of order $(k/M)^2$, and hence negligible in the energy range of interest here. We shall therefore not include line currents, but imitate their effect by subtracting the difference of our nongauge-invariant calculation and the low-energy theorem.

Relativistic dispersion relations have also been used to investigate this problem.[7] Such an approach has the advantage of not requiring a cutoff and allowing in principle the inclusion of nucleon recoil effects. On the other hand these treatments suffer from the fact that in practice they are essentially a "one-meson approximation" and furthermore require some guesses about the high-energy behavior of the matrix elements. It is hoped that the static theory as developed in this paper will shed some light on the validity of the various assumptions used in a dispersion relation approach.

II. CALCULATION

We wish to evaluate the Compton cross section using second-order perturbation theory in the electromagnetic coupling. The unperturbed system is to be the meson-nucleon system, in the static approximation. Then the unperturbed Hamiltonian is the usual one of the Chew[1] theory, namely:

$$H=\sum \omega_\kappa a_\kappa{}^\dagger a_\kappa+\sum_\kappa(V_\kappa a_\kappa+V_\kappa{}^\dagger a_\kappa{}^\dagger). \quad (1)$$

The notation is standard. κ is an index denoting the momentum $\boldsymbol{\kappa}$ and charge state $\kappa=1, 2, 3$ of a meson. Furthermore,

$$\omega_\kappa=(\boldsymbol{\kappa}^2+\mu^2)^{\frac{1}{2}},$$

and

$$V_\kappa=(4\pi)^{\frac{1}{2}}\left(\frac{if_0}{\mu}\right)\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\kappa}}{(2\omega_\kappa)^{\frac{1}{2}}}\tau_\kappa,$$

where f_0 is the unrenormalized coupling constant, and μ is the meson mass. $\boldsymbol{\sigma}$ and τ_κ are spin and isotopic spin operators for the nucleon. The unperturbed eigenstates, that is, the eigenstates of H, are denoted by $\psi_n^{(-)}$, with energy E_n. The superscript $(-)$ is used to indicate the eigenfunction with "incoming wave boundary conditions."[1] In addition the unperturbed Hamiltonian should include the Hamiltonian of the free radiation field. This will be suppressed.

The perturbing Hamiltonian describes the interaction of the electromagnetic field and the meson nucleon system. This will contain two parts, linear and quadratic in the charge e. Both will include contributions from line currents. According to the philosophy which we have adopted for handling line currents, however, we do not need an explicit form for them. We must merely remember to subtract the two lowest terms in the photon energy from the expression we get ignoring line currents, and then insert the low-energy theorem. This method of treating the line currents corresponds to the "subtractions" which are made in the dispersion theory approach to this problem.[7] The perturbing Hamiltonian neglecting line currents is then all we need to consider. This is

$$H_{\text{pert}}=-\int \mathbf{j}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})d^3x+e^2\int\phi^*(\mathbf{x})\phi(\mathbf{x})d^3x+\frac{e^2}{2M}A^2(0). \quad (2)$$

Here $\mathbf{A}(\mathbf{x})$ is the photon field operation, $\phi(\mathbf{x})$ is the meson field operator, and $\mathbf{j}(\mathbf{x})$ is the total current operator (except for line currents) of the meson-nucleon system. These operators are all in the Schrödinger representation. The last term in Eq. (2) represents the amplitude to scatter from a static nucleon, and gives the Thomson cross section.

The matrix element of interest is now obtained by doing first order perturbation theory in the terms in e^2, and second order perturbation theory in the $\mathbf{j}\cdot\mathbf{A}$ term. Explicitly, we wish to calculate:

$$M(\mathbf{ke}\to\mathbf{k}'\mathbf{e}')=2\frac{e^2}{2M}[\mathbf{A}'(0)]^*\cdot\mathbf{A}(0)+2e^2\int[\mathbf{A}'(\mathbf{x})]^*\cdot\mathbf{A}(\mathbf{x})\langle\psi_0|\phi^*(\mathbf{x})\phi(\mathbf{x})|\psi_0\rangle$$
$$+\sum_n\frac{\langle\psi_0|-\int\mathbf{j}\cdot\mathbf{A}'^*|\psi_n^{(-)}\rangle\langle\psi_n^{(-)}|-\int\mathbf{j}\cdot\mathbf{A}|\psi_0\rangle}{k-E_n+i\epsilon}-\sum_n\frac{\langle\psi_0|-\int\mathbf{j}\cdot\mathbf{A}|\psi_n^{(-)}\rangle\langle\psi_n^{(-)}|-\int\mathbf{j}\cdot\mathbf{A}'^*|\psi_0\rangle}{k+E_n}. \quad (3)$$

[6] F. E. Low, Phys. Rev. **96**, 1428 (1954); M. Gell-Mann and M. L. Goldberger, Phys. Rev. **96**, 1433 (1954).

[7] R. H. Capps (private communication); J. Mathews (private communication); J. Mathews and M. Gell-Mann (to be published).

In this equation the matrix elements of the photon field have been explicitly evaluated, and thus

$$\mathbf{A}(\mathbf{x})=\frac{1}{(2k)^{\frac{1}{2}}}\mathbf{e}e^{i\mathbf{k}\cdot\mathbf{x}},$$

$$\mathbf{A}'(\mathbf{x})=\frac{1}{(2k)^{\frac{1}{2}}}\mathbf{e}'e^{i\mathbf{k}'\cdot\mathbf{x}},$$

where $\mathbf{ke}$ and $\mathbf{k'e'}$ are the momenta and polarizations of the initial and final photons. We have also taken $k=k'$; that is, we have assumed energy conservation.

This is perhaps an appropriate place to discuss briefly the kinematics involved in the static model. Under the assumption that the nucleon is infinitely heavy, on which the static theory is based, there is no distinction between, for example, the laboratory and center-of-mass coordinate systems. The calculation is done in either system, and in both the nucleon is at rest before and after the collision. In actuality, of course, the nucleon is not infinitely heavy, so the system used cannot really exist. The question then is whether this fictitious system best approximates the real laboratory system or the real center-of-mass system, or perhaps something else. The fact that $k=k'$ in the fictitious system and that $k=k'$ also in the real center-of-mass system indicates that the fictitious system used in static model calculations should be identified with the actual center-of-mass system. This will be done here.

One further comment should be made. If line current effects are introduced into Eq. (3), making the theory gauge-invariant, it is easy to show explicitly in a manner similar to that used by Capps[5] in a perturbation (in f^2) calculation of the process, that the Thomson limit actually does result at zero photon energy.

We now turn to the explicit evaluation of the matrix element $M(\mathbf{ke}\rightarrow\mathbf{k'e'})$. We shall restrict ourselves to electric dipole and magnetic dipole scattering, and shall discuss the electric dipole effect first. Consider the terms in Eq. (3) involving the current. This current consists of an interaction current and a meson current:

$$\mathbf{j}(\mathbf{x})=(4\pi)^{\frac{1}{2}}\left(\frac{ief_0}{\mu}\right)\boldsymbol{\sigma}[\tau_3\boldsymbol{\tau}\cdot\boldsymbol{\phi}(\mathbf{x})-\boldsymbol{\tau}\cdot\boldsymbol{\phi}(\mathbf{x})\tau_3] \\ -ie[\phi(\mathbf{x})\nabla\phi^*(\mathbf{x})-\phi^*(\mathbf{x})\nabla\phi(\mathbf{x})]. \quad (4)$$

It may also be decomposed in a different way.[1] We write

$$\mathbf{j}=\mathbf{j}_\pi+\mathbf{j}_s+\mathbf{j}_v, \quad (5)$$

where

$$\langle\psi_0|\mathbf{j}_\pi|\psi_0\rangle=0,$$

and

$$[a_p,\mathbf{j}_s]=[a_p,\mathbf{j}_v]=0.$$

It is easily seen from parity conservation that $\mathbf{j}_s$ and $\mathbf{j}_v$ contribute only to magnetic dipole Compton scattering. For the present, therefore, we shall replace $\mathbf{j}$ by $\mathbf{j}_\pi$.

Consider the matrix element

$$\left\langle\psi_n^{(-)}\middle|\int\mathbf{j}_\pi\cdot\mathbf{A}\middle|\psi_0\right\rangle.$$

This vanishes for $n=0$, by definition of $\mathbf{j}_\pi$. The sum on n in Eq. (3) therefore runs from one to infinity instead of zero to infinity. The matrix element, then, represents single photoproduction ($n=1$), double photoproduction ($n=2$), triple photoproduction ($n=3$), etc. If the photon which is absorbed here is not magnetic dipole or electric quadrupole, one of the final mesons (namely that one which coupled to the photon) cannot be a P-wave meson. It therefore cannot interact with the nucleon, so we may write

$$\psi_n^{(-)}=a_p^\dagger\psi_{n-1}^{(-)} \quad (6)$$

where p is the non-P-wave meson which absorbed the photon. In general, for any multipole photon, we have

$$\psi_n^{(-)}=a_p^\dagger\psi_{n-1}^{(-)}-\frac{1}{H-E_n-i\epsilon}V_p\psi_{n-1}^{(-)}, \quad (7)$$

and hence

$$\left\langle\psi_n^{(-)}\middle|-\int\mathbf{j}_\pi\cdot\mathbf{A}\middle|\psi_0\right\rangle=\left\langle\psi_{n-1}^{(-)}\middle|\left[a_p,\int\mathbf{j}_\pi\cdot\mathbf{A}\right]\middle|\psi_0\right\rangle \\ -\left\langle\psi_{n-1}^{(-)}\middle|V_p^\dagger\frac{1}{H-E_n-i\epsilon}\int\mathbf{j}_\pi\cdot\mathbf{A} \right. \\ \left. +\int\mathbf{j}_\pi\cdot\mathbf{A}\frac{1}{H+\omega_p}V_p^\dagger\middle|\psi_0\right\rangle. \quad (8)$$

As indicated above, the second term here contributes only to magnetic dipole and electric quadrupole scattering. For present purposes we therefore restrict ourselves to the first term. We note that $\mathbf{j}_\pi$ may be replaced by $\mathbf{j}$ here, by virtue of Eq. (5).

The commutator is readily evaluated. We obtain

$$\left\langle\psi_{n-1}^{(-)}\middle|\left[a_p,\int\mathbf{j}_\pi\cdot\mathbf{A}\right]\middle|\psi_0\right\rangle=-\sum_{\alpha=1}^{3}\epsilon_{\alpha p3}\frac{ie}{(2k)^{\frac{1}{2}}}\frac{2\mathbf{e}\cdot\mathbf{p}}{(4\omega_p\omega_{\mathbf{p}-\mathbf{k}})^{\frac{1}{2}}}\delta_{n-1;\,k-p,\,\alpha} \\ +\sum_{\alpha=1}^{3}\epsilon_{\alpha p3}\frac{ie}{(2k)^{\frac{1}{2}}}\left\langle\psi_{n-1}^{(-)}\middle|(4\pi)^{\frac{1}{2}}\frac{if}{\mu}\frac{\boldsymbol{\sigma}\cdot\mathbf{l}(E_{n-1},p\mathbf{ke})}{(2\omega_p)^{\frac{1}{2}}}v(|\mathbf{p}-\mathbf{k}|)\tau_\alpha\middle|\psi_0\right\rangle, \quad (9)$$

where we have defined

$$\mathbf{l}(E_n,p\mathbf{ke})=\mathbf{e}+\frac{2\mathbf{e}\cdot\mathbf{p}(\mathbf{p}-\mathbf{k})}{E_n^2-\omega_{\mathbf{p}-\mathbf{k}}^2+i\epsilon}. \quad (10)$$

The symbol $\delta_{n-1;\,k-p,\,\alpha}$ vanishes unless the state $n-1$ is a one-meson state, containing a meson of momentum $\mathbf{k}-\mathbf{p}$ and charge α. In the following discussion, we shall be interested in two vectors $\mathbf{l}$; those for the initial and final photons. For convenience in writing, these two vectors will be denoted $\mathbf{l}(E_n)$ and $\mathbf{l}'(E_n)$. Also, $\mathbf{L}$ shall be used for $\mathbf{l}$ with the sign of $\mathbf{k}$ changed.

The $\sum_n$ terms of Eq. (3) have then been reduced to

$$-\sum_{n=1}^{\infty}\left[\frac{\langle\psi_0|[a_p,\int\mathbf{j}\cdot\mathbf{A}']^\dagger|\psi_{n-1}{}^{(-)}\rangle\langle\psi_{n-1}{}^{(-)}|[a_p,\int\mathbf{j}\cdot\mathbf{A}]|\psi_0\rangle}{E_{n-1}+\omega_p-k-i\epsilon}+\frac{\langle\psi_0|[a_p,\int\mathbf{j}\cdot\mathbf{A}^*]^\dagger|\psi_{n-1}{}^{(-)}\rangle\langle\psi_{n-1}{}^{(-)}|[a_p,\int\mathbf{j}\cdot\mathbf{A}'^*]|\psi_0\rangle}{E_{n-1}+\omega_p+k}\right]. \tag{11}$$

No approximations have been made in obtaining this result for any multipole other than magnetic dipole or electric quadrupole. Several diagrams included in this equation are shown as examples in Fig. 1(a).

The $\sum_n$ in Eq. (11) may now be evaluated by making use of the relation

$$\frac{f}{\mu^2}\sum_{n=1}^{\infty}\langle\psi_0|\sigma_i\tau_{\alpha'}|\psi_n{}^{(-)}\rangle\langle\psi_n{}^{(-)}|\sigma_j\tau_\alpha|\psi_0\rangle\delta(E_n-\omega)=\frac{4}{\kappa^3v^2(\kappa)}\sum_{JT}\sin^2\delta_{JT}(\omega)\langle i|P_J|j\rangle\langle\alpha'|Q_T|\alpha\rangle, \tag{12}$$

where P_J and Q_T are the usual angular momentum and isotopic spin projection operations, defined by

$$\begin{gathered}\langle i|P_{1/2}|j\rangle=(1/4\pi)\sigma_i\sigma_j,\quad \langle i|P_{3/2}|j\rangle=(1/4\pi)(3\delta_{ij}-\sigma_i\sigma_j),\\ \langle\alpha'|Q_{1/2}|\alpha\rangle=\tfrac{1}{3}\tau_{\alpha'}\tau_\alpha,\quad \langle\alpha'|Q_{3/2}|\alpha\rangle=\tfrac{1}{3}(3\delta_{\alpha'\alpha}-\tau_{\alpha'}\tau_\alpha).\end{gathered} \tag{13}$$

Replacing $n-1$ by n in Eq. (11), separating out the $n=0$ term, and employing Eq. (12), we finally obtain for the $\sum_n$ terms of Eq. (3) the result:

$$\begin{aligned}
-\frac{e^2}{2k}\left(\frac{4\pi f^2}{\mu^2}\right)\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega_p}&\left\{\frac{\boldsymbol{\sigma}\cdot\left(\mathbf{e}'-\dfrac{2\mathbf{e}'\cdot\mathbf{p}(\mathbf{p}-\mathbf{k}')}{\omega_{\mathbf{p}-\mathbf{k}'}{}^2}\right)\boldsymbol{\sigma}\cdot\left(\mathbf{e}-\dfrac{2\mathbf{e}\cdot\mathbf{p}(\mathbf{p}-\mathbf{k})}{\omega_{\mathbf{p}-\mathbf{k}}{}^2}\right)}{\omega_p-k-i\epsilon}\right.\\
&\left.+\frac{\boldsymbol{\sigma}\cdot\left(\mathbf{e}-\dfrac{2\mathbf{e}\cdot\mathbf{p}(\mathbf{p}+\mathbf{k})}{\omega_{\mathbf{p}+\mathbf{k}}{}^2}\right)\boldsymbol{\sigma}\cdot\left(\mathbf{e}'-\dfrac{2\mathbf{e}'\cdot\mathbf{p}(\mathbf{p}+\mathbf{k}')}{\omega_{\mathbf{p}+\mathbf{k}'}{}^2}\right)}{\omega_p+k}\right\}-\frac{4e^2}{3k}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega_p}\int\frac{d\omega_\kappa\sin^2\delta(\kappa)}{\kappa^3v^2(\kappa)}\\
&\times\left\{\frac{2\mathbf{l}'^*(\omega_\kappa)\cdot\mathbf{l}(\omega_\kappa)-i\boldsymbol{\sigma}\cdot\mathbf{l}'^*(\omega_\kappa)\times\mathbf{l}(\omega_\kappa)}{\omega_\kappa+\omega_p-k-i\epsilon}+\frac{2\mathbf{L}^*(\omega_\kappa)\cdot\mathbf{L}'(\omega_\kappa)-i\boldsymbol{\sigma}\cdot\mathbf{L}^*(\omega_\kappa)\times\mathbf{L}'(\omega_\kappa)}{\omega_\kappa+\omega_p+k}\right\}+\frac{4\pi^2e^2}{2k}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\omega_p}\\
&\times\left\{\frac{\sin(\delta(|\mathbf{p}-\mathbf{k}'|))e^{i\delta(|\mathbf{p}-\mathbf{k}'|)}}{\omega_{\mathbf{p}-\mathbf{k}'}|\mathbf{p}-\mathbf{k}'|^3}\frac{\mathbf{e}'\cdot\mathbf{p}}{\omega_{\mathbf{p}-\mathbf{k}'}+\omega_p-k-i\epsilon}[2(\mathbf{k}'-\mathbf{p})\cdot\mathbf{l}(\omega_{\mathbf{p}-\mathbf{k}'})-i\boldsymbol{\sigma}\cdot(\mathbf{k}'-\mathbf{p})\times\mathbf{l}(\omega_{\mathbf{p}-\mathbf{k}'})]\right.\\
&\left.+\frac{\sin(\delta(|\mathbf{p}+\mathbf{k}|))e^{i\delta(|\mathbf{p}+\mathbf{k}|)}}{\omega_{\mathbf{p}+\mathbf{k}}|\mathbf{p}+\mathbf{k}|^3}\left(\frac{\mathbf{e}\cdot\mathbf{p}}{\omega_{\mathbf{p}+\mathbf{k}}+\omega_p+k-i\epsilon}\right)[-2(\mathbf{k}+\mathbf{p})\cdot\mathbf{l}(\omega_{\mathbf{p}+\mathbf{k}})+i\boldsymbol{\sigma}\cdot(\mathbf{k}+\mathbf{p})\times\mathbf{l}(\omega_{\mathbf{p}+\mathbf{k}})]\right\}\\
&+\text{the complex conjugate of the entire last integral with the interchanges } \mathbf{k}\leftrightarrow\mathbf{k}',\ \mathbf{e}\leftrightarrow\mathbf{e}'.
\end{aligned} \tag{14}$$

In deriving this result we have dropped the term proportional to $\delta_{n-1;\,\mathbf{k}-\mathbf{p},\,\alpha}\delta_{n-1;\,\mathbf{k}'-\mathbf{p},\,\alpha}$ since this is a vacuum polarization effect and does not give a contribution to the scattering proper. (See Fig. 2.)

The three terms in Eq. (14) arise from the following sources. The first comes from the $n-1=0$ term of Eq. (11) and represents single photoproduction of a meson $\mathbf{p}$ followed by radiative capture of $\mathbf{p}$. The second comes from the sum of all the other terms, with $\boldsymbol{\sigma}\cdot\mathbf{l}$ from each commutator. The last results from the cross terms between a $\boldsymbol{\sigma}\cdot\mathbf{l}$ from one commutator and a $\delta_{n-1;\,\mathbf{k}-\mathbf{p},\,\alpha}$ from the other. It therefore appears only in the $n-1=1$ term of the sum. Equation (14) is an exact expression for the scattering of all multipoles except $M1$ and $E2$, together with some $M1$ and $E2$ scattering. As it stands, it is an unpleasantly complicated expression, which simplifies, however, on picking out just the electric dipole amplitude. This amplitude may be extracted using the appropriate electric dipole projection operator.

Denote the electric dipole amplitude of the first term of Eq. (14) by

$$(e^2/2Mk)[A_1(k)\mathbf{e}'\cdot\mathbf{e}+B_1(k)i\boldsymbol{\sigma}\cdot\mathbf{e}'\times\mathbf{e}],$$

and of the second and third terms of Eq. (14) by

$$(e^2/2Mk)[A_2(k)\mathbf{e}'\cdot\mathbf{e}+B_2(k)i\boldsymbol{\sigma}\cdot\mathbf{e}'\times\mathbf{e}].$$

We first discuss A_1 and B_1; that is, the first term in Eq. (14). Physically, this represents the single photoproduction of a meson followed by its radiative capture, together with the crossed analog of this. Consider the expression

$$P(\mathbf{p}\leftarrow\mathbf{k}\hat{e})\equiv(4\pi)^{\frac{1}{2}}\left(\frac{ief}{\mu}\right)\frac{1}{(2k\omega_p)^{\frac{1}{2}}}\times\boldsymbol{\sigma}\cdot\left(\mathbf{e}-\frac{2\mathbf{e}\cdot\mathbf{p}(\mathbf{p}-\mathbf{k})}{\omega_{\mathbf{p}-\mathbf{k}}{}^2}\right). \quad (15)$$

This is the matrix element, in perturbation theory, to photoproduce a positive meson $\mathbf{p}$ from a photon $\mathbf{k}\hat{e}$. The electric dipole part of this can be written:

$$P_{E1}(\mathbf{p}\leftarrow\mathbf{ke})=k^{-\frac{1}{2}}[f_S\boldsymbol{\sigma}\cdot\mathbf{e}+f_D(\boldsymbol{\sigma}\cdot\mathbf{e}-3P^{-2}\boldsymbol{\sigma}\cdot\mathbf{pp}\cdot\mathbf{e})]. \quad (16)$$

Here f_S and f_D are the amplitudes to produce S and D wave mesons, respectively; they are functions of k^2 and $\omega_p{}^2$. Note that since $k\neq\omega_p$, this amplitude is off the energy shell and hence does not describe a physical process. For the case $k=\omega_p$, we have as the S- and D-wave electric dipole photoproduction cross sections

$$\sigma_S(\omega_p)=\frac{p}{\pi}|f_S(k=\omega_p)|^2,\quad \sigma_D(\omega_p)=\frac{2p}{\pi}|f_D(k=\omega_p)|^2. \quad (17)$$

It should be noted that the exact prediction of the static theory for f_S and f_D is obtained by evaluating them from Eq. (15). This is again because only P-wave mesons can couple to the nucleon in this model.

Using Eqs. (15) and (16), we see that

$$\frac{e^2}{2Mk}A_1(k)=-\frac{1}{\pi^2k}\int p\omega_p{}^2d\omega_p\left(\frac{|f_S|^2+2|f_D|^2}{\omega_p{}^2-k^2-i\epsilon}\right);$$
$$\frac{e^2}{2Mk}B_1(k)=-\frac{1}{\pi^2}\int p\omega_pd\omega_p\left(\frac{|f_S|^2-|f_D|^2}{\omega_p{}^2-k^2-i\epsilon}\right). \quad (18)$$

A_1 and B_1 could therefore be easily evaluated directly from this expression. It is of interest, however, to transform these equations a bit further. Suppose for the moment that we ignore the meson current. Then

$$f_S=(4\pi)^{\frac{1}{2}}\left(\frac{ief}{\mu}\right)\frac{1}{(2\omega_p)^{\frac{1}{2}}}\quad\text{and}\quad f_D=0.$$

Furthermore,

$$\sigma_S=\frac{2e^2f^2}{\mu^2}\left(\frac{p}{\omega_p}\right).$$

We therefore obtain

$$\frac{e^2}{2Mk}A_1(k)=-\frac{1}{\pi k}\int\omega_p{}^2d\omega_p\left(\frac{\sigma_S(\omega_p)}{\omega_p{}^2-k^2-ie}\right),$$

and (19)

$$\frac{e^2}{2Mk}B_1(k)=-\frac{1}{\pi}\int\omega_pd\omega_p\left(\frac{\sigma_S(\omega_p)}{\omega_p{}^2-k^2-i\epsilon}\right).$$

Performing the subtractions to take care of the line currents, we have the amplitude of interest:

$$\frac{e^2}{2Mk}[A_1(k)-A_1(0)-kA_1'(0)]=-\frac{k}{\pi}\int d\omega_p\left(\frac{\sigma_S(\omega_p)}{\omega_p{}^2-k^2-i\epsilon}\right),$$
$$\frac{e^2}{2Mk}[B_1(k)-B_1(0)-kB_1'(0)]=-\frac{k^2}{\pi}\int\frac{d\omega_p}{\omega_p}\left(\frac{\sigma_S(\omega_p)}{\omega_p{}^2-k^2-i\epsilon}\right). \quad (20)$$

Then equations are seen to be identical in form with the dispersion relations of Gell-Mann and Mathews.[7] The only differences are (i) that they have in principle (though not in practice) included recoil and (ii) that they use experimental values for $\sigma_S(\omega_p)$ whereas we, to be consistent, should use the predictions of the static theory.

If the meson current is retained, the identification of our Eqs. (18) with the electric dipole dispersion relations is not quite so immediate, because the amplitudes f_S and f_D in Eq. (18) are off the energy shell. By using the (by now) standard arguments for deriving dispersion relations for a static potential, for example, it is seen that the off-energy shell amplitudes in Eq. (18) can be put on the energy shell. Thus Eqs. (20) hold with the meson current included as well (with a slight modification for the presence of D waves).

As stated above, the consistent procedure for us to follow would be to use the static theory to calculate σ_S and σ_D. This is easily done since perturbation theory is correct for S- and D-wave mesons. This model agrees fairly well for low energies; however, it does not do so well for high energies. A better numerical result would therefore presumably be obtained if we inserted experimental values for σ_S and σ_D. We have done this.

The remaining part of Eq. (14), namely A_2 and B_2, can be evaluated in a similar way by using the electric dipole projection operators. These terms correspond to multiple meson production, followed by inverse multiple meson production. The final expressions for A_2 and B_2 are quite lengthy and will not be reproduced; we shall content ourselves with a brief summary of the numerical results. A_2 turns out to be almost independent

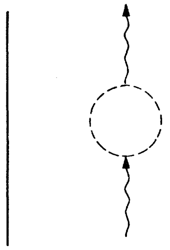

FIG. 2. Vacuum polarization diagram appearing in Compton effect.

of k, up to at least $k=2\mu$. Its actual value is about 1.8, and it varies by less than 1%. B_2 is not constant, but it remains very small; its maximum value in the range 0 to 2μ is about 0.05. Once the subtractions are performed, then, A_2 and B_2 will effectively have vanished. This reinforces very strongly the assumption made in the dispersion-theoretic approach that higher order intermediate states are unimportant.

Next discuss the term in $\phi^*\phi$ in Eq. (3). This may also be evaluated exactly in terms of the meson nucleon scattering phase shifts. In order to do this, we follow the technique described by Fubini.[8] The desired matrix element is

$$\frac{e^2}{k}\mathbf{e}'\cdot\mathbf{e}\int d^3x\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\langle\psi_0|\phi^*(\mathbf{x})\phi(\mathbf{x})|\psi_0\rangle. \quad (21)$$

Upon expanding the meson field into creation and destruction operators and introducing the quantities $R(\omega)$ defined by Fubini,[8] this may be rewritten as

$$\frac{4\pi e^2}{k}\mathbf{e}'\cdot\mathbf{e}\int\frac{d^3\kappa}{(2\pi)^3}\int\frac{d^3\kappa'}{(2\pi)^3}\frac{v(\kappa)v(\kappa')}{\omega_\kappa^2-\omega_{\kappa'}^2}\boldsymbol{\kappa}\cdot\boldsymbol{\kappa}$$
$$\times\int d^3x\,\exp[i(\boldsymbol{\kappa}+\boldsymbol{\kappa}'+\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}]$$
$$\times\left\{\frac{R_1(\omega_\kappa)}{\omega_\kappa}-\frac{R_1(\omega_{\kappa'})}{\omega_{\kappa'}}\right\}, \quad (22)$$

where if we, for example, neglect all meson-nucleon phase shifts except that arising from the 33 states,

$$R_1(\omega)=\frac{f^2}{\mu^2\omega}+\frac{4}{3\pi}\int\frac{d\omega_{\kappa'}}{\kappa'^3v^2(\kappa')}\left(\frac{\sin^2\delta_{33}(\omega_{\kappa'})}{\omega+\omega_{\kappa'}}\right). \quad (23)$$

This contains all multipoles. We again wish to select only the $E1$ amplitude; denote it by

$$(e^2/2Mk)A_3(k)\mathbf{e}'\cdot\mathbf{e}.$$

Note that there is no spin-flip contribution from this term. We obtain

$$A_3(k)=64\pi^2M\int\int\int\frac{\kappa^3d\kappa\,\kappa'^3d\kappa'}{\omega_\kappa^2-\omega_{\kappa'}^2}j_1(\kappa x)j_1(\kappa' x)$$
$$\times\left[\frac{R_1(\omega_\kappa)}{\omega_\kappa}-\frac{R_1(\omega_{\kappa'})}{\omega_{\kappa'}}\right][j_0^2(kx)+j_2^2(kx)]x^2dx. \quad (24)$$

[8] S. Fubini, Nuovo cimento **3**, 1425 (1956).

The space integral here is laborious but straightforward.[9] The remaining integrals on κ and κ' are done numerically. The result shows $A_3(k)$ to be a decreasing function of k, starting at about eight Thomson amplitudes at $k=0$, with zero slope.

The final electric dipole scattering comes from the last term in Eq. (3), representing the scattering due to a point static nucleon. The amplitude resulting from this is just the usual Thomson term, $(e^2/2Mk)\mathbf{e}'\cdot\mathbf{e}$.

Altogether, the electric dipole scattering amplitude is given by

$$(e^2/2Mk)\{\mathbf{e}'\cdot\mathbf{e}[A_1(k)+A_2(k)+A_3(k)+1]$$
$$+i\boldsymbol{\sigma}\cdot\mathbf{e}'\times\mathbf{e}[B_1(k)+B_2(k)]\}. \quad (25)$$

The foregoing analysis has produced the exact prediction of the static theory for the electric dipole scattering amplitude, barring line current effects. We now turn our attention to magnetic dipole scattering. Here, unfortunately, no complete calculation seems possible, due to the fact that the same meson need not interact with both photons. Thus diagrams such as those shown in Fig. 3 can contribute in this case. We are, however, helped by two circumstances. First, the

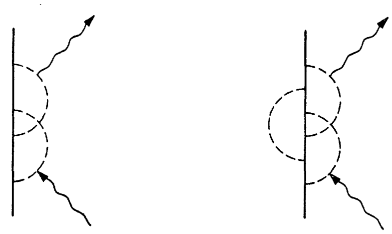

FIG. 3. Diagrams in which different mesons interact with the photons, thus contributing only to $M1$ and $E2$ scattering.

magnetic dipole scattering is fairly small at low energies compared to the electric dipole, so that an accurate calculation should not be necessary. Second, at high energies, the entire scattering is dominated by a large resonance in the magnetic dipole, and the resonance terms can be calculated exactly.

Magnetic dipole scattering will arise from the following terms. The $\mathbf{j}_s$ and $\mathbf{j}_v$ parts of the current contribute only for $M1$ photons. The second term in Eq. (7) gives an $M1$ effect. In addition, these are $M1$ effects from the terms already calculated from which we earlier extracted only the electric dipole contribution. Of all these sources of $M1$ scattering, only that coming from $\mathbf{j}_v$ is important at high energies. This part produces a huge resonance at about 300 Mev in the lab system, and completely swamps the other effects. At low energies, $\mathbf{j}_v$ does not dominate, but our subtractions and use of the low-energy theorem should repair most of the damage produced by using $\mathbf{j}_v$ alone. Furthermore, as stated above, the $M1$ scattering is small at low energies, so approximating it to some extent does no harm.

[9] G. N. Watson, *Bessel Functions* (Cambridge University Press, New York, 1944), second edition, chap. XIII.

We therefore limit ourselves to $\mathbf{j}_v$. We have[1]

$$\left\langle \psi_n^{(-)} \left| \int \mathbf{j}_v \cdot \mathbf{A} \right| \psi_0 \right\rangle = \left(\frac{\mu_p - \mu_n}{2}\right) \frac{\mu}{(4\pi f)^{\frac{1}{2}}} \times \left\langle \psi_n^{(-)} \left| \frac{\boldsymbol{\sigma} \cdot \mathbf{k} \times \mathbf{e}}{(2k)^{\frac{1}{2}}} \tau_3 \right| \psi_0 \right\rangle. \quad (26)$$

Inserting this in $\sum_n$ in Eq. (3), we notice that the result is identical to the integral equation satisfied by the amplitude to scatter a neutral meson of momentum $\mathbf{k}\times\mathbf{e}$ into one of momentum $\mathbf{k}'\times\mathbf{e}'$, barring certain factors. Correcting these factors, the exact result for the magnitude dipole scattering from $\mathbf{j}_v$ may be easily written down. It is, if we retain only the 33 phase shift

$$-\tfrac{1}{3}\left(\frac{\mu_p - \mu_n}{2f}\right)^2 \frac{\mu^2}{kq^3} \sin\delta_{33}(k) e^{i\delta_{33}(k)} \times [2(\mathbf{k}'\times\mathbf{e}')\cdot(\mathbf{k}\times\mathbf{e}) - i\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{e}')\times(\mathbf{k}\times\mathbf{e})], \quad (27)$$

where $q^2 = k^2 - \mu^2$. The small phase shifts could also be included, but their effect is only about 10% even near $k=\mu$, so they will be neglected. We write this amplitude as

$$(e^2/2Mk)k^{-2}\{\mathbf{k}'\times\mathbf{e}'\cdot\mathbf{k}\times\mathbf{e}C_1(k) + i\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{e}')(\mathbf{k}\times\mathbf{e})[-\tfrac{1}{2}C_1(k)]\}.$$

The low-energy theorem states that the Compton scattering amplitude, including terms in $1/k$ and independent of k, is given by[6]

$$M_0 = \frac{e^2}{2Mk}\mathbf{e}'\cdot\mathbf{e} + i\mu^2 k^{-2}\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{e}')\times(\mathbf{k}\times\mathbf{e}) + \frac{ie\mu}{2M}\boldsymbol{\sigma}\cdot\left[\left\{\frac{\mathbf{k}(\mathbf{k}\times\mathbf{e})+(\mathbf{k}\times\mathbf{e})\mathbf{k}}{2k^2}\right\}\cdot\hat{e}' - \left\{\frac{\mathbf{k}'(\mathbf{k}'\times\mathbf{e}')+(\mathbf{k}'\times\mathbf{e}')\mathbf{k}'}{2k^2}\right\}\cdot\hat{e}\right] - (ie\mu_A/2M)\boldsymbol{\sigma}\cdot\mathbf{e}'\times\mathbf{e}, \quad (28)$$

where μ is the total magnetic moment of the nucleon, and μ_A its anomalous part. If we pick out of this the $E1$ and $M1$ contributions, we are left with just

$$M_0 = \frac{e^2}{2Mk}\mathbf{e}'\cdot\mathbf{e} - \frac{e\mu_A}{2M} i\boldsymbol{\sigma}\cdot\mathbf{e}'\times\mathbf{e} + \mu^2 k^{-2}\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{e}')\times(\mathbf{k}\times\mathbf{e}). \quad (29)$$

Our complete scattering amplitude then takes its final form:

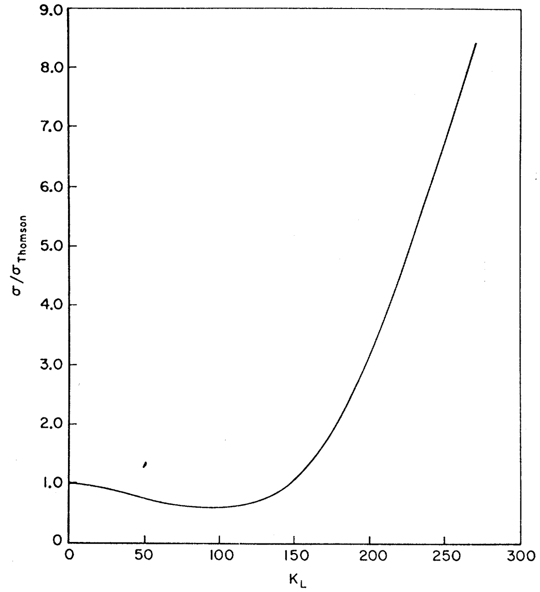

FIG. 4. The total cross section including both $E1$ and $M1$ effects.

$$\begin{aligned} M(\mathbf{ke}\to\mathbf{k}'\mathbf{e}')_{\text{complete}} = (e^2/2Mk)\{&\mathbf{e}'\cdot\mathbf{e}[1+A_1(k) - A_1(0) - kA_1'(0) + A_2(k) - A_2(0) - kA_2'(0) \\ &+ A_3(k) - A_3(0) - kA_3'(0)] + i\boldsymbol{\sigma}\cdot\mathbf{e}'\times\mathbf{e}[-(k\mu_A/e) \\ &+ B_1(k) - B_1(0) - kB_1'(0) + B_2(k) - B_2(0) - kB_2'(0)] \\ &+ k^{-2}(\mathbf{k}'\times\mathbf{e}')\cdot(\mathbf{k}\times\mathbf{e})[C_1(k) - C_1(0) - kC_1'(0)] \\ &+ ik^{-2}\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{e}')\times(\mathbf{k}\times\mathbf{e})[(2Mk\mu^2/e^2) - \tfrac{1}{2}C_1(k) \\ &+ \tfrac{1}{2}C_1(0) + \tfrac{1}{2}kC_1'(0)]\}. \quad (30)\end{aligned}$$

This includes subtractions to reproduce the line current, and the insertion of the low-energy theorem. Setting A_2, B_2, A_3, and C_1 equal to zero gives us the dispersion relation of Gell-Mann and Mathews for electric dipole scattering in the static limit.

III. RESULTS

The differential cross section for the Compton scattering process is readily obtained from Eq. (30), and may be written as

$$\begin{aligned} d\sigma/d\Omega = (e^2/4\pi M)^2[&(\tfrac{1}{2}|A|^2 + \tfrac{1}{2}|C|^2 + \tfrac{3}{2}|B|^2 + \tfrac{3}{2}|D|^2) \\ &+ (\tfrac{1}{2}|A|^2 + \tfrac{1}{2}|C|^2 - \tfrac{1}{2}|B|^2 - \tfrac{1}{2}|D|^2)\cos^2\theta \\ &+ (AC^* + A^*C + BD^* + B^*D)\cos\theta], \quad (31)\end{aligned}$$

where A, B, C, and D are defined as the coefficients of $\mathbf{e}'\cdot\mathbf{e}$, $i\boldsymbol{\sigma}\cdot(\mathbf{e}'\times\mathbf{e})$, $k^{-2}(\mathbf{k}'\times\mathbf{e}')\cdot(\mathbf{k}\times\mathbf{e})$, and $ik^{-2}\boldsymbol{\sigma}\cdot(\mathbf{k}'\times\mathbf{e}')\times(\mathbf{k}\times\mathbf{e})$, respectively in Eq. (30). The numerical evaluation of the various amplitudes making up A, B, C, and D has already been briefly discussed. The total cross section [obtained by integrating Eq. (31)] is plotted in Fig. 4 as a function of incident photon energy in the laboratory system. In Fig. 5 we show the electric dipole contribution to this cross section (that is, we set C and D equal to zero). The differential cross section for a center-of-mass scattering angle of 90° is plotted

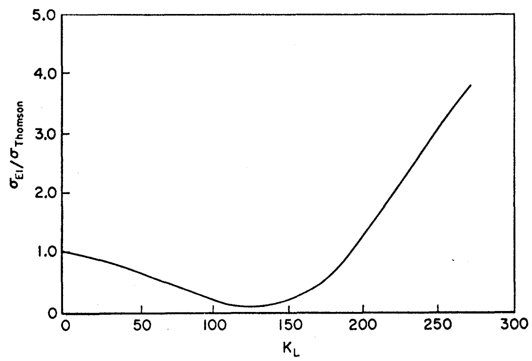

FIG. 5. The electric dipole contribution to the total cross section.

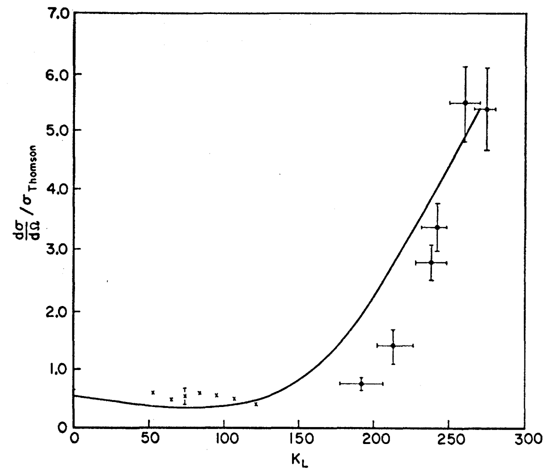

FIG. 6. $d\sigma/d\Omega$ at 90° in the center-of-mass system. Experimental points from reference 10.

in Fig. 6, along with experimental data.[10] At energies larger than about 1.5 meson rest masses, the cross section is pretty well dominated by the resonance in the magnetic dipole $j=\frac{3}{2}$ state. At low energies, the cross section is predominantly electric dipole. The dip appearing in the cross section is due to a cancellation between electric dipole scattering from the meson cloud and the Thomson amplitude. It is seen that the experiments are qualitatively reproduced, but the agreement is not ideal, particularly around 200 Mev. It should be mentioned that the size of the magnetic dipole contribution, which is what is responsible for the larger than desirable values near 200 Mev, is quite sensitive to the choice of scattering phase shifts. Since these are not too well known in this region, this may be an explanation of the disagreement.

[10] Yamagata, Auerbach, Bernardini, Fuosofo, Hansen, and Odian, Bull. Am. Phys. Soc. Ser. II, 1, 350 (1956).

IV. ACKNOWLEDGMENTS

We wish to thank M. Gell-Mann and J. Mathews for a number of interesting and useful discussions on the relation of the dispersion theoretic approach to this problem to the present work. One of us (WKRW) would like to thank the National Science Foundation for a research grant.